A Note on the Dimensions of the Structural Invariant Subspaces of the Discrete-Time Singular Hamiltonian Systems

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Abstract

The structural invariant subspaces of the discrete-time singular Hamiltonian system are used in [1] to give an analytic nonrecursive expression of all the admissible trajectories. A deeper insight into the features of these subspaces, particularly focused on the dimensionality issue, is the object of this note.

I. Main Content

In [1], the structural invariant subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$, defined by (17) and (18) respectively, are used in (19), that is the analytic nonrecursive expression of the set of the admissible solutions of the discrete-time singular Hamiltonian system (10) over the time interval $0 \leq k \leq k_f - 1$.

In this note it will be shown that the dimension of the subspace $\mathcal{V}_2$ may be lower than n, where n denotes the dimension of the state space of the original system as defined in (1), (2). In particular, the possible loss of dimension of $\mathcal{V}_2$ (or, equivalently, the possible loss of rank of the matrix V_2 defined by (18)) depends on the properties of the original system (1), (2) (under assumptions $\mathcal{A}.1$–$\mathcal{A}.4$).

Let us consider system (1), (2) and perform the similarity transformation $T = [T_1 \; T_2]$, where $\operatorname{im} T_1 = \mathcal{R}$, the reachable subspace of (A, B). With respect to the new basis,

$$A = \begin{bmatrix} A_c & A_{cu} \\ O & A_u \end{bmatrix}, \; B = \begin{bmatrix} B_c \\ O \end{bmatrix},$$

$$C = \begin{bmatrix} C_c & C_u \end{bmatrix}, \quad D = D.$$

Moreover, the solution P_+ of the Riccati equation (11), (12), partitioned accordingly, is

$$P_+ = \begin{bmatrix} P_c & P_{cu} \\ P_{cu}^\top & P_u \end{bmatrix},$$

where P_c is the stabilizing solution of the Riccati equation restricted to the sole reachable part of the original system: i.e.,

$$P_c = A_c^\top P_c A_c + C_c^\top C_c - (A_c^\top P_c B_c + C_c^\top D)(D^\top D + B_c^\top P_c B_c)^{-1}(B_c^\top P_c A_c + D^\top C_c),$$

with

$$D^\top D + B_c^\top P_c B_c > 0.$$

The stabilizing feedback is partitioned as

$$-K_+ = \begin{bmatrix} K_c & K_u \end{bmatrix}.$$

Similarly, the solution W of the discrete Lyapunov equation has the structure

$$W = \begin{bmatrix} W_c & O \\ O & O \end{bmatrix},$$

where W_c is the solution of the discrete Lyapunov equation restricted to the sole reachable part of the original system: i.e.,

$$(A_c + B_c K_c) W_c (A_c + B_c K_c)^\top + B_c (D^\top D + B_c^\top P_c B_c)^{-1} B_c^\top = W_c.$$

Simple algebraic manipulations, where these partitions are taken into account, yield the following structure for the matrix V_2:

$$V_2 = \begin{bmatrix} W_c (A_c + B_c K_c)^\top & O \\ O & O \\ \star & O \\ \star & -A_u^\top \\ \star & O \end{bmatrix},$$

where the symbol $\star$ denotes a possibly nonzero submatrix. The structure pointed out in the partitioned matrix V_2 shows that the rank of V_2 may be lower than n. This circumstance occurs, for instance, if A_u has a zero row, like in the illustrative example considered in the following section.

It is worth noting that, by contrast, the subspace

$$\bar{\mathcal{V}}_2 = \operatorname{im} \begin{bmatrix} W \\ P_+ W - I \end{bmatrix},$$

that is used in (20) of [1] in order to express the sole state and costate trajectories over the time interval $0 \leq k \leq k_f$, has dimension n.

In fact, the corresponding partitioned matrix is

$$\bar{V}_2 = \begin{bmatrix} W_c & O \\ O & O \\ \star & O \\ \star & -I \end{bmatrix}.$$

The rank of $\bar{V}_2$ is n, since the symmetric positive definite W_c, being the solution of the restricted Lyapunov equation above, has the same rank of the controllability Gramian of the pair $(A_c + B_c K_c, B_c)$, which is completely controllable by construction.

II. An Illustrative Example

This section presents a numerical example where the rank of matrix V_2 is lower than the dynamic order n of the original system, while n is the rank of matrix $\bar{V}_2$. The variables are displayed in scaled fixed point format with five digits, although computations are made in floating point precision. Consider system (1), (2) in [1], with

$$A = \begin{bmatrix} 0.3 & -0.4 & 0.5 & 0.6 \\ 0.1 & 0.2 & 0.1 & 0.1 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0.2 \\ 2 & 3 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 5 & 6 \end{bmatrix}, \quad D = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix},$$

By pursuing the procedure illustrated in [1], one gets, in particular

$$V_2 = \begin{bmatrix} 0.3661 & -0.4314 & 0 & 0 \\ -0.7323 & 0.8629 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0000 & 0.0000 & 0 & 0 \\ 0.0000 & -0.0000 & 0 & 0 \\ 1.8443 & -1.1885 & -0.5000 & 0 \\ -0.0000 & 0.0000 & 0 & 0 \\ 0.1098 & -0.1294 & 0 & 0 \\ -0.2088 & 0.4374 & 0 & 0 \end{bmatrix},$$

and

$$\bar{V}_2 = \begin{bmatrix} 0.4708 & -0.5165 & 0 & 0 \\ -0.5165 & 1.1828 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.0000 & 0.0000 & 0 & 0 \\ 0.0000 & -0.0000 & 0 & 0 \\ 3.6885 & -2.3770 & -1.0000 & 0 \\ 2.7295 & 0.5411 & 0 & -1.0000 \end{bmatrix}.$$

III. Conclusions

In this note, it has been shown that the dimension of the structural invariant subspace $\mathcal{V}_2$ may be lower than the dynamic order n of the original system. The result has been illustrated by a numerical example. This is the reason why the only-if part of the proof of Theorem 1 in [1] does not rely on a dimensionality count, but on the maximality of the subspace $\mathcal{V}_2$ (and of the subspace $\mathcal{V}_1$). In fact, maximality follow from Property 2 (and Property 1, respectively), according to [2, Section 5.4].

References

[1] E. Zattoni, "Structural invariant subspaces of singular Hamiltonian systems and nonrecursive solutions of finite-horizon optimal control problems," *IEEE Transactions on Automatic Control*, vol. 53, no. 5, pp. 1279–1284, June 2008.

[2] V. Ionescu, C. Oară, and M. Weiss, *Generalized Riccati Theory and Robust Control: A Popov Function Approach*. Chichester, England: John Wiley & Sons, 1999.